Math 261B Thurs. 10/1

SO_N preserves form with matrix $J = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$ $\quad k^N \quad e_1, \dots, e_N$

dual basis $e_N, \dots, e_1$

$\Downarrow$

$A \qquad x^T J y = x^T A^T J A y \qquad A^T J A = J$

$JA^TJA = 1$

$\;'' A^R$

$A^R A = I$

$\det A^R = \det A$

$A^R A = I$

$\Rightarrow \det(A)^2 = 1$

Lie algebra and $\det(A) = 1$

$A \mapsto A^R$

k alg

$p \in T$

$k[\varepsilon]/(\varepsilon^2) = k \oplus k\varepsilon$

$k[x]/(x-a)(x-b)$

$\downarrow$

$k[x]/(x^2)$

$$\begin{array}{ccc} T & \to & X \\ \Downarrow & & \Downarrow \\ P & \to & Q \end{array}$$

$\mathcal{O}(X) \to k[\varepsilon]/(\varepsilon^2)$

m_Q^2

m_Q

$k[\varepsilon]/(\varepsilon) \;''\; K$

(ε)

Morphisms $T \to X$

$= K$ alg homs $\mathcal{O}(X) \to k[\varepsilon]/(\varepsilon^2)$

$= (Q \in X, \; v \in T_Q X)$

$\;'' (m_Q/m_Q^2)^*$

$A = I + \varepsilon M \qquad \varepsilon \in k[\varepsilon]/\varepsilon^2$

$SO_N \subset SL_N \qquad gl_N = M_N \qquad so_N \subset gl_N$

$$(I+\varepsilon M)^R (I+\varepsilon M) - I = (I+\varepsilon M^R)(I+\varepsilon M) - I$$

$$= I + \varepsilon M^R + \varepsilon M - I = \varepsilon(M^R + M)$$

so_N is $M^R + M = 0$

$$M^R = -M \qquad \begin{pmatrix} & x & & 0 \\ & & \cdot^{\cdot^{\cdot}} & -x \\ 0 & & & \end{pmatrix}$$

(char $k = 2$: don't get 0's on diagonal)

SO_{2n+1}

$$T = \begin{pmatrix} t_1 & & & & & & \\ & t_2 & & & & & \\ & & \ddots & & & & \\ & & & t_n & & & \\ & & & & 1 & & \\ & & & & & t_n^{-1} & \\ & & & & & & \ddots \\ & & & & & & & t_1^{-1} \end{pmatrix}$$

$T^R T = I \qquad T^R = T^{-1}$

$T \subset B \subset SO_{2n+1}$

diagonal, upper Δ'ar

so_{2n+1}

$$\begin{array}{cc} & \begin{array}{ccc} & j & \end{array} \\ i \rightarrow & \begin{pmatrix} & x & 0 \\ 0 & & -x \\ & & 0 \\ 0 & & \end{pmatrix} \end{array} \leftarrow 2n+2-j$$

$\uparrow$ $2n+2-i$

Conjugate by $\underline{t} \in T$ $\quad$:

$t_i/t_j\ x$

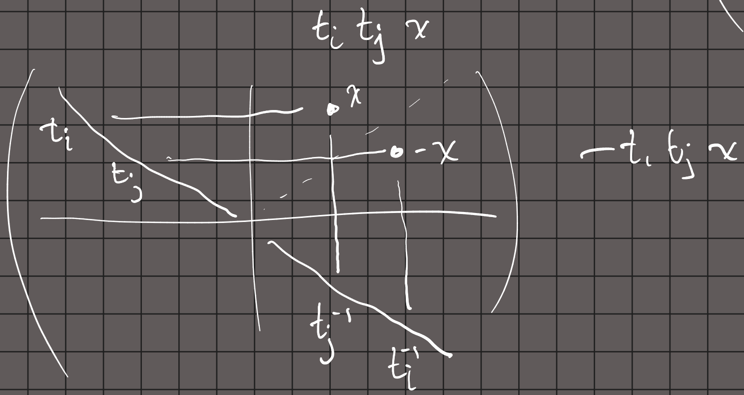
$$\left(\begin{array}{c|c} t_i \;—\; x & \\ t_j & \\ \hline t_j^{-1} & —\; -x \\ & t_i^{-1} \end{array}\right)$$

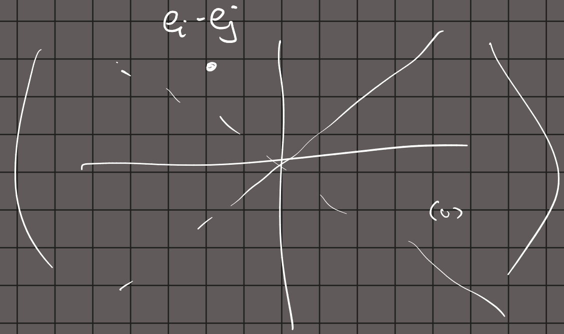
$-t_j^{-1}/t_i^{-1}\ x$

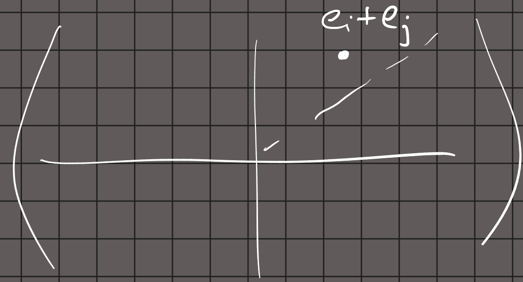
$= -t_i/t_j\ x$

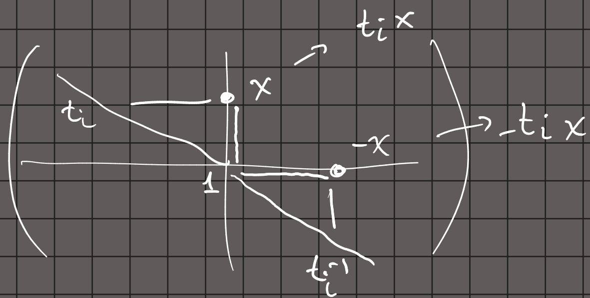
$X(T) \cong \mathbb{Z}^n$

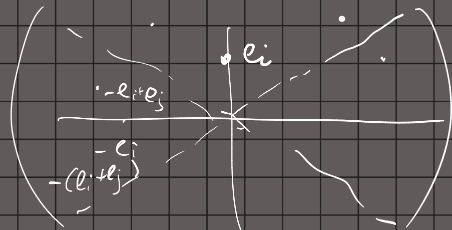
$\lambda \in \mathbb{Z}^n \mapsto (\underline{t} \mapsto \underline{t}^\lambda)$











$R \subset X$ is $\{\pm e_i \pm e_j \mid i \neq j\} \cup \{\pm e_i\}$ $\quad Q = \mathbb{Z} \cdot R = X$

SO_{2n+1} is "adjoint": center is trivial, acts faithfully on its Lie alg.

Root SL_2's

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} 1 & & & & & & & \\ & a & b & & & & & \\ & c & d & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & a & -b & \\ & & & & & -c & d & \\ & & & & & & & 1 \end{pmatrix}$$

i, j

$A \in SL_2$

$A \qquad A' = A^R$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$\det(A) = 1$

$$\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$ is an automorphism of SL_2

$$\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & & x & & \\ & \ddots & & -x & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \qquad U_\alpha \quad \alpha = e_i - e_j$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} t & a & & b & \\ & a\,t & & & -b \\ & & & & \\ c & & t^{-1} & d & \\ & -c & & & d\,t^{-1} \end{pmatrix}$$

$U_\alpha \quad \alpha = e_i + e_j \qquad \varepsilon_i + \varepsilon_j$

$X = \mathbb{Z}^n \qquad X^* \cong \mathbb{Z}^n$

$$\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \mapsto \begin{pmatrix} 1 & & & & & & \\ & t & & & & & \\ & & t^{-1} & & & & \\ & & & \ddots & & & \\ & & & & t & & \\ & & & & & t^{-1} & \\ & & & & & & 1 \end{pmatrix}$$

$t_i = t \qquad t_k = t_k^0 \qquad \varepsilon_i - \varepsilon_j$

$t_j = t^{-1}$

$\alpha = \pm e_i \pm e_j \quad \longleftrightarrow \quad \alpha^\vee = \pm \varepsilon_i \pm \varepsilon_j \qquad \langle \alpha^\vee, \alpha \rangle = 2$

$\alpha = e_i$

$$\begin{pmatrix} 1 & x & -x^2/2 ?? \\ & 1 & -x \\ & & 1 \end{pmatrix} \quad e_i$$

$SL_2 \to SO_3$

$\Vert$

$PGL_2 = PSL_2$

$SL_2 / \{\pm I\}$

$\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \mapsto$

SL_2 has a standard irr. rep V_2 with weights $2, 0, -2$

dim = 3

$x^2 \quad xy \quad y^2$

$SL_2 \overset{Ad}{\curvearrowright} sl_2$

dim = 3

$\uparrow$

Conjugation by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ weight 2 $\qquad \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \cdot E \mapsto t^{(2)} E$

$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ weight 0

$F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ weight -2

$e_1 \rightleftarrows e_2$

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad [E, F] = H$

$\langle\ ,\ \rangle$ on sl_2 is $tr(XY) = tr(YX)$ $\quad tr(EF) = 1$

$tr\, H^2 = 2$

$$\begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$$

$tr(HE) = 0 = tr(HF)$

$tr(E^2) = tr(F^2)$

$tr(H/\sqrt{2}, H/\sqrt{2}) = 1$

$E,\ H/\sqrt{2},\ F$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} E \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -c & a \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -ac & a^2 \\ -c^2 & ac \end{pmatrix} = a^2 E - \sqrt{2}\, ac \frac{H}{\sqrt{2}} - c^2 F$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} H \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ c & -a \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} ad+bc & -2ba \\ 2cd & -(ad+bc) \end{pmatrix} = -\sqrt{2}\, ba\, E$$

$(ad+bc)\, H/\sqrt{2}$

$\sqrt{2}\, cd F$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} F \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 0 \\ d & -b \end{pmatrix} = \begin{pmatrix} bd & -b^2 \\ d^2 & -bd \end{pmatrix} = -b^2 E + \sqrt{2}\, bd\, H/\sqrt{2}$$

$+ d^2 F$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a^2 & -\sqrt{2}\, ba & -b^2 \\ -\sqrt{2}\, ac & \frac{ad+bc}{2} & \sqrt{2}\, bd \\ -c^2 & \sqrt{2}\, cd & d^2 \end{pmatrix} = A \qquad A^R A = I$$

$$\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \mapsto$$

$$\begin{pmatrix} 1 & -\sqrt{2}x & -x^2 \\ & 1 & \sqrt{2}x \\ & & 1 \end{pmatrix}$$

$$\begin{pmatrix} \ddots & \vdots & & \vdots & \\ & a^2 & -\sqrt{2}ba & -b^2 & \\ & -\sqrt{2}ac & \frac{ad+bc}{2} & \sqrt{2}bd & \\ & c^2 & \sqrt{2}cd & d^2 & \\ & & & & \ddots \end{pmatrix}$$

$(SL_2)_\alpha$ for $\alpha = e_i$

$a = t$

$d = t^{-1}$

$$\begin{pmatrix} \ddots & & & \\ & t^2 & & \\ & & 1 & \\ & & & t^{-2} \\ & & & & \ddots \end{pmatrix}$$

$\mathbb{G}_m \to T$

t

$t_i = t^2$

Root $\alpha = e_i \longleftrightarrow$ coroot $\alpha^\vee = 2\varepsilon_i$

$\langle \alpha^\vee, \alpha \rangle = 2$ $\quad 2\varepsilon_i$

Weyl group:

$\alpha^\vee = \varepsilon_i - \varepsilon_j$ $\quad \alpha = e_i - e_j$ $\quad s_\alpha$ is transposition $e_i \leftrightarrow e_j$

$\alpha^\vee = 2\varepsilon_i$ $\quad \alpha = e_i$ $\quad s_\alpha$ is $e_i \longleftrightarrow -e_i$

W = Signed permutations $= S_n \ltimes \{\pm 1\}^n$

$\varepsilon_i + \varepsilon_j$ $\quad e_i + e_j$ $\quad e_i \leftrightarrow -e_j$

Pos. roots R_+ are $e_i \pm e_j$ $i<j$ and e_i

Simple roots Δ: $\underset{\alpha_1}{e_1-e_2}, \dots, \underset{\alpha_{n-1}}{e_{n-1}-e_n}, \underset{\alpha_n}{e_n}$ $\qquad \alpha_n^\vee = 2\varepsilon_n$

$\langle \varepsilon_2-\varepsilon_3, e_1-e_2\rangle$

$$e_i - e_j = e_i - e_{i+1} + \dots + e_{j-1}-e_j$$

$$e_i \qquad e_i - e_n + e_n$$

$$e_i+e_j \qquad e_i - e_j + 2e_j$$

Coxeter group generators of W: transpositions $(i,i+1)$, s_i, $i=1,\dots,n-1$ — S_n; $\quad e_n \mapsto -e_n$, s_n

$\langle \alpha_j^\vee, \alpha_i\rangle$

$$\begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & \ddots & & & \\ & & & 2 & -1 & \\ & & & -1 & 2 & -2 \\ & & & & -1 & 2 \end{pmatrix}$$

$\circ - \circ - \circ - \cdots - \circ \Rightarrow \circ$

B_n

$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ $\quad \circ \quad \circ$

$\langle 2\varepsilon_n, e_{n-1}-e_n\rangle$

$\langle \varepsilon_{n-1}-\varepsilon_n, e_n\rangle$

$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ $\quad \circ - \circ$

$\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$ $\quad \circ \Rightarrow \circ$